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Discussion paper

Investment under Vanishing Uncertainty Due to Information Arriving over Time

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Abstract

In this paper a new market model is considered. A firm has the opportunity to invest in a project of uncertain profitability. Over time, the firm gets – stochastically – signals on the profitability of the project. The belief that the firm needs to have in a profitable project for investment to be optimal is calculated and analyzed. It is shown that the probability of investing in a project with low profitability is larger when the firm uses the conventional net present value rule.

Keywords: Market uncertainty, Optimal investment, Monopolistic behaviour.
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1 Introduction

In recent years, investment theory has experienced a new branch of literature. It is concerned with taking into account the option value of waiting in evaluating investment projects. The seminal work by Dixit and Pindyck

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(1994) illustrates this idea by using methods well-known in the financial markets literature (cf. Duffie (1996)). The key-idea is that when a risky project is irreversible and not a "once in a lifetime" opportunity, then postponing investment until further information becomes available creates an option value of waiting. The investment is undertaken if the value of investing immediately exceeds the value of waiting for more information. One of the first papers to consider the option value of waiting in evaluating an investment project is McDonald and Siegel (1986).

This paper applies the option idea to a potential monopolist who faces the decision whether or not to invest in a project. He has incomplete information about the state of the world, which is given by the conditions in the relevant market, in the sense that it is not known beforehand whether or not it is profitable to enter this market. The market being profitable or not can be thought of as being driven for example by uncertainty about the interest of consumers in the product the monopolist would produce. One can think about the telecommunication sector where there is one company that considers to supply a new service to its customers. However, the company is uncertain about the possible succes of the new service. Therefore, the company issues a market investigation that occasionally reports about the market expectations.

This situation is modelled by considering a market that can be either good or bad. If the market is bad, the optimal strategy is to refrain from investment. Since the monopolist incurs sunk costs when investing in the project, a loss is suffered in case the market is bad. Occasionally however, the monopolist receives a signal about the quality of the market. The moments in time at which signals arrive are unknown beforehand. Therefore, by delaying investment and waiting for more signals to arrive, the monopolist can predict with higher probability whether the market is good or bad. This induces an option value of waiting. The question is how many good signals relative to bad signals the monopolist needs, to make it optimal to invest in the project. We will see that this is equivalent to finding a critical level for the belief that the market is good, given the available signals.

The signals can be thought of as nature flipping a coin that with a certain probability gives the true conditions of the market. As soon as the monopolist has invested, the true state of the world is revealed. The decision problem we consider in this paper can therefore be seen as a game with incomplete information, where at time zero, nature chooses the state of the world. Through time, the incompleteness of the information resolves because of the signals that arrive.

It is important to note that our model differs fundamentally from Dixit and Pindyck (1994). Whereas our model is a game with incomplete information where nature determines the state of the world only at the beginning of the game, the framework presented in Dixit and Pindyck (1994) is a game with complete information, where nature determines the state of the world at each consecutive point in time.

This difference can have important implications. Consider for instance antitrust policy. As Dixit and Pindyck (1994, ch. 9) correctly argue, antitrust authorities should consider the uncertainty, that influences the decision of a firm concerning entering a market. Therefore, the antitrust authorities should consider the barriers that uncertainty raises and not confuse them with barriers that firms may develop to reduce entry by others and hence reduce competition. In the Dixit and Pindyck argument however, it is not taken into account that the uncertainty that a possible entrant faces, is likely to reduce over time. So, where standard Marshallian analysis would encourage authorities to act too soon against, in their view, excessive prices, the Dixit and Pindyck approach might induce them to wait too long before measures are taken.

Our approach is related to Jensen (1982). The main difference is that in Jensen's model, the true state of the project is never completely revealed. The probability of a good project is an unknown parameter. Initially, this probability is unknown. In each period one receives a signal about the true value of the unknown parameter. This signal is used to update the beliefs, just as in our model, i.e. the belief is a conditional probability based on past information. In short, one forms a belief on the belief in a good project. However, in Jensen's model, a good signal not only increases the belief in a good project, but it also increases the probability of receiving a good signal next period. In other words, Jensen implicitly assumes that nature might flip a biased coin. So, the quality of the signals increases if more good signals are received. The magnitude of the increase in the quality of the signal equals the magnitude of the increase in the belief in a good project. Especially this last feature might be considered to be too strong an assumption. In our model we circumvent these problems by assuming that the quality of the signal is independent of past realizations, i.e. the investor exactly knows the quality of the coin that nature flips. Contrary to Jensen (1982), the analysis of our framework provides an explicit expression for the critical value of the belief in a good project at which investing is optimal.

The paper is organised as follows. In Section 2 the formal model is described. After that, the traditional net present value approach and the optimal approach will be extensively discussed in Section 3. In Section 4 the results obtained in Section 3 will be interpreted using a numerical example. In the final section some conclusions will be drawn.

2 The Model

Consider a monopolist who faces the choice of investing in a certain project. The market conditions can be either good in the sense that demand is high, leading to high revenues, denoted by U^H, or bad when low demand leads to low revenues U^L.[1][2] Without loss of generality we assume $U^L = 0$. The sunk costs involved in investing in the project are given by $I > 0$.

It is assumed that when the monopolist receives the option to invest, he does have a prior belief about the market being good or bad. The *ex ante* probability of high revenues is given by

$$\mathbb{P}(\mathrm{H}) = \mathrm{p}_0.$$

Occasionally, the monopolist receives a signal revealing the market to be good (denoted by h) or a signal revealing the market to be bad (denoted by l). The probabilities with which these signals occur depend on the situation of the market. A correct signal is given with probability $\lambda > \frac{1}{2}$, see Table 1. As soon as the monopolist invests in the project, the state of the market is revealed. In reality this may take some time, but we abstract from that. The signals' arrivals are modelled via a Poisson process with parameter $\mu > 0$. The Poisson assumption is made to make the model analytically tractible

[1] These revenues may result from different demand and/or cost functions depending on the state of the world. See Section 4 for an example.

[2] The revenues represent an infinite cash flow discounted at rate r.

	h	l
H	λ	$1-\lambda$
L	$1-\lambda$	λ

Table 1: Probabilities, where the first row lists the probabilities in case of high demand and the second row in case of low demand.

when using dynamic programming. Hence, denoting the number of signals by n, this boils down to

$$dn(t) = \begin{cases} 1 & \text{with probability } \mu dt, \\ 0 & \text{with probability } 1-\mu dt, \end{cases} \tag{1}$$

with

$$n(0) = 0. \tag{2}$$

Denoting the number of h-signals by g, the dynamics of g is then given by,

$$dg(t) = udn(t), \tag{3}$$

with

$$u = \begin{cases} 1 & \text{with probability } \lambda \text{ if } H \text{ and } 1-\lambda \text{ if } L, \\ 0 & \text{with probability } 1-\lambda \text{ if } H \text{ and } \lambda \text{ if } L, \end{cases}$$

and

$$g(0) = 0. \tag{4}$$

For notational convenience the time indices will be suppressed in the remainder of the paper. The belief that revenues are high, i.e. that the market is good, given the number of signals n and the number of h-signals $g \leq n$ is denoted by $p(n,g)$. Now, the conditional expected payoff of the monopolist can be written as,

$$\mathrm{I\!E}(\mathrm{U}|\mathrm{n},\mathrm{g}) = \mathrm{p}(\mathrm{n},\mathrm{g})(\mathrm{U}^{\mathrm{H}} - \mathrm{I}) + (1 - \mathrm{p}(\mathrm{n},\mathrm{g}))(\mathrm{U}^{\mathrm{L}} - \mathrm{I}). \tag{5}$$

The structure of the model is such that with respect to the signals there are two main aspects. The first one is the parameter which governs the arrival of the signals, μ. This parameter is a measure for the quantity of the signals, since $1/\mu$ denotes the average time between two signals. The other component is the probability of the correctness of the signal, λ. This parameter is a measure for the quality of the signals. For the model to make sense, it is assumed that $\lambda > \frac{1}{2}$. This assumption is not as strong as it seems, for f $\lambda < \frac{1}{2}$ the monopolist can perform the same analysis replacing λ with $1 - \lambda$. If $\lambda = \frac{1}{2}$ the signals are not informative at all and the monopolist would do best by making a now-or-never decision, using his *ex ante* belief $p(0,0) = p_0$. In this paper learning – or belief extracting – takes place by using the Bayesian approach. This, together with the condition $\lambda > \frac{1}{2}$, implies that the belief in high revenues converges to one or to zero if the market is good or bad, respectively, in the long-run. As will be shown in Section 3.2 quantity and quality together determine the threshold belief in a good market that the monopolist needs to have to be willing to invest.

3 Analysis

In determining the optimal output level, the monopolist chooses the output that maximizes his expected profit flow. Since we assume the monopolist to be risk-neutral, he is only interested in the expected values of investing in the project and waiting for more information. Furthermore we assume that the monopolist discounts all future profits at a rate r.

The expected payoff given the number of signals n and the number of h-signals g is given by,

$$\begin{aligned} \mathbb{E}(\mathrm{U|n,g}) &= p(n,g)(U^H - I) + (1 - p(n,g))(-I) \\ &= U^H p(n,g) - I. \end{aligned} \tag{6}$$

3.1 The Net Present Value Approach

Under the net present value (NPV) approach, the monopolist is indifferent between investing and not investing if $\mathbb{E}(\mathrm{U|n,g})$ equals zero. If it is smaller than zero he will abstain from investing and if it is larger than zero he will invest immediately. Hence, there is a critical level for $p(n,g)$, which we denote by p_{NPV}, at which the monopolist is indifferent. This critical level is given by,

$$p_{NPV} = \frac{I}{U^H}. \tag{7}$$

If the monopolist believes that, given the number of signals and the number of good signals, the probability that revenues are high equals p_{NPV}, then, under the NPV approach, he is indifferent between investing and not investing. This implies that there is a critical level of combinations of n and g that make the monopolist indifferent. In other words, the monopolist needs enough good signals relative to the number of bad signals to invest in the project.

Concerning the critical level p_{NPV} there are two possibilities. Either $p_{NPV} > 1$ or $p_{NPV} \leq 1$.[3] In the former case, it is never optimal for the monopolist to invest. The latter case holds if

$$U^H \geq I \tag{8}$$

This is a straightforward condition, for if it were the case that the monopolist would not invest even if he knew that revenues are high for sure, then it is also never optimal to invest if there is uncertainty about the market conditions.

3.2 The Optimal Investment Decision

As stated in the introduction, the uncertainty about the market conditions and the irreversibility of the entrance to the market induce an option value of waiting for more signals. In this subsection we will show how to find the critical level for $p(n, g)$ at which the firm is indifferent between investing and waiting, while taking into account the option value of waiting.

First, we explicitly calculate $p(n, g)$. To simplify matters considerably,

[3] Note from eq. (7) that it always holds that $p_{NPV} \geq 0$.

define $k := 2g - n$ and $\zeta := \frac{1-p_0}{p_0}$. We now obtain using Bayes' rule,

$$\begin{aligned} p(n,g) &= \frac{\mathbb{P}(\mathrm{n,g|H})\mathbb{P}(\mathrm{H})}{\mathbb{P}(\mathrm{n,g|H})\mathbb{P}(\mathrm{H}) + \mathbb{P}(\mathrm{n,g|L})\mathbb{P}(\mathrm{L})} \\ &= \frac{\lambda^g(1-\lambda)^{n-g}p_0}{\lambda^g(1-\lambda)^{n-g}p_0 + (1-\lambda)^g\lambda^{n-g}(1-p_0)} \\ &= \frac{\lambda^{2g-n}p_0}{\lambda^{2g-n}p_0 + (1-\lambda)^{2g-n}(1-p_0)} \\ &= \frac{\lambda^k}{\lambda^k + \zeta(1-\lambda)^k} \equiv p(k). \end{aligned} \tag{9}$$

The critical level of k where the monopolist is indifferent between investing and not investing in the project is denoted by k^*. Note that at any arrival of an h-signal k increases with one, and at any arrival of an l-signal k decreases with one. Hence, enough h-signals must arrive to reach the critical level. The critical level of the conditional belief of high revenues is denoted by $p^* = p(k^*)$.

Suppose that the state of the process at a particular point in time is given by k. Then there are three possibilities. First, k might be such that $k \geq k^*$ and $p(k) \geq p^*$. Then it is optimal for the monopolist to directly invest in the project.[4] In this case the value for the monopolist, denoted by Ω, equals the expected payoff given in eq. (6):

$$\Omega(k) = U^H p(k) - I. \tag{10}$$

A second possibility is that, even after a new h-signal arriving, it is still not optimal to invest, i.e. $k < k^* - 1$. The value of the opportunity to invest for the monopolist, denoted by V_1, can then be constructed from the following second order linear difference equation,

$$\begin{aligned} V_1(k) = {} & e^{-rdt}\{(1-\mu dt)V_1(k) + \mu dt[p(k)(\lambda V_1(k+1) + (1-\lambda)V_1(k-1)) + \\ & + (1-p(k))(\lambda V_1(k-1) + (1-\lambda)V_1(k+1))]\} \\ = {} & (1-rdt)(1-\mu dt)V_1(k) + (1-rdt)\mu dt[(2p(k)\lambda + 1 - \lambda - p(k)) \\ & V_1(k+1) + (p(k) + \lambda - 2p(k)\lambda)V_1(k-1)] + o(dt) \\ \Leftrightarrow \quad & (r+\mu)dtV_1(k) = \mu dt[(2p(k)\lambda + 1 - \lambda - p(k))V_1(k+1) + \\ & + (p(k) + \lambda - 2p(k)\lambda)V_1(k-1)] + o(dt). \end{aligned} \tag{11}$$

[4] This happens if $k^* < k_0 = 0$.

Eq. (11) states that the value of the option to enter at state k must equal the discounted expected value a small amount of time later. It is nothing else than using the Bellman principle for the continuation region[5]. Using eq. (9) it holds that

$$2p(k)\lambda + 1 - \lambda - p(k) = \frac{\lambda^{k+1} + \zeta(1-\lambda)^{k+1}}{\lambda^k + \zeta(1-\lambda)^k} \tag{12}$$

and

$$\begin{aligned} p(k) + \lambda - 2p(k)\lambda &= \frac{\lambda^k - \lambda^{k+1} + \lambda\zeta(1-\lambda)^k}{\lambda^k + \zeta(1-\lambda)^k} \\ &= \frac{\lambda(1-\lambda)(\lambda^{k-1} + \zeta(1-\lambda)^{k-1})}{\lambda^k + \zeta(1-\lambda)^k}. \end{aligned} \tag{13}$$

Substituting eqs. (12) and (13) into eq. (11) yields, after dividing by dt, letting dt approach zero and rearranging terms, the Bellman equation:

$$\begin{aligned} (r+\mu)(\lambda^k + \zeta(1-\lambda)^k)V_1(k) =& \mu(\lambda^{k+1} + \zeta(1-\lambda)^{k+1})V_1(k+1) \\ &+ \mu\lambda(1-\lambda)(\lambda^{k-1} + \zeta(1-\lambda)^{k-1})V_1(k-1). \end{aligned} \tag{14}$$

After defining $F(k) := (\lambda^k + \zeta(1-\lambda)^k)V_1(k)$, eq. (14) can be written as

$$(r+\mu)F(k) = \mu F(k+1) + \mu\lambda(1-\lambda)F(k-1). \tag{15}$$

Eq. (15) is a second order linear homogeneous difference equation which has as general solution

$$F(k) = A\beta^k, \tag{16}$$

where A is a constant and β is a solution of the homogeneous equation,

$$\mathcal{Q}(\beta) \equiv \beta^2 - \frac{r+\mu}{\mu}\beta + \lambda(1-\lambda) = 0. \tag{17}$$

Eq. (17) has two real roots,[6] namely

$$\beta_{1,2} = \frac{r+\mu}{2\mu} \pm \tfrac{1}{2}\sqrt{(\tfrac{r}{\mu}+1)^2 - 4\lambda(1-\lambda)}. \tag{18}$$

Note that $\mathcal{Q}(0) = \lambda(1-\lambda) > 0$ and $\mathcal{Q}(1-\lambda) = -\frac{r}{\mu}(1-\lambda) \leq 0$. Since the graph of $\mathcal{Q}$ is an upward pointing parabola we must have $\beta_1 \geq 1-\lambda$ and $0 < \beta_2 < 1-\lambda$ (see Figure 1). The value function V_1 is then given by

[5]The continuation region is the region where it is optimal to refrain from investing and thus wait for more information.

[6]It should be noted that for all λ it holds that $4\lambda(1-\lambda) \leq 1$. Since equality holds iff $\lambda = 1/2$, the homogeneous equation indeed has two real roots for any $\lambda \in (1/2, 1]$.

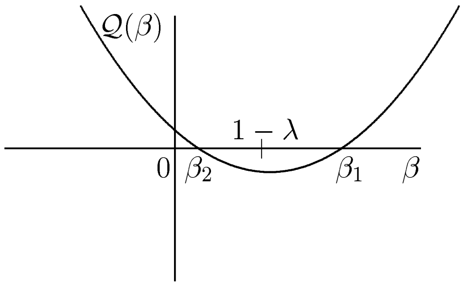

Figure 1: Graph of $\mathcal{Q}$.

$$V_1(k) = \frac{F(k)}{\lambda^k + \zeta(1-\lambda)^k} = \frac{A_1\beta_1^k + A_2\beta_2^k}{\lambda^k + \zeta(1-\lambda)^k}. \tag{19}$$

The last case is the case where k is such that it is not optimal to invest in the project right away. However, if the following signal indicates high revenues, it will then be optimal to invest, i.e $k^* - 1 \leq k < k^*$. In this region the value function V_2 for the monopolist must satisfy the following Bellman equation:

$$rV_2 = \tfrac{1}{dt}\mathbf{E}(dV_2). \tag{20}$$

Calculating eq. (20) yields,

$$\begin{aligned} rV_2(k) &= \mu[p(k)(\lambda\Omega(k+1) + (1-\lambda)V_1(k-1)) + (1-p(k)) \\ &\quad (\lambda V_1(k-1) + (1-\lambda)\Omega(k+1)) - V_2(k)] \\ &= \mu[(2p(k)\lambda + 1 - \lambda - p(k))\Omega(k+1) + \\ &\quad + (p(k) + \lambda - 2p(k)\lambda)V_1(k-1) - V_2(k)] \\ \Leftrightarrow \quad & (r+\mu)V_2(k) = \mu[(2p(k)\lambda + 1 - \lambda - p(k))\Omega(k+1) + \\ &\quad + (p(k) + \lambda - 2p(k)\lambda)V_1(k-1)]. \end{aligned} \tag{21}$$

Substituting eqs. (19), (10), (12) and (13) into eq. (21) yields

$$\begin{aligned} V_2(k) =& \frac{\mu}{r+\mu}\Big(\lambda U^H p(k) - (\lambda p(k) + (1-\lambda)(1-p(k)))I \\ & + \lambda(1-\lambda)\tfrac{A_1\beta_1^{k-1} + A_2\beta_2^{k-1}}{\lambda^k + \zeta(1-\lambda)^k}\Big). \end{aligned} \tag{22}$$

If an h-signal arrives, the process jumps to the region where $k \geq k^*$ and if an l-signal arives the process jumps to the region where $k < k^*$. Therefore the value V_2 is completely determined by $V_1(k-1)$ and $\Omega(k+1)$. The value function V is then given by (cf. eqs. (10), (19) and (22))

$$V(k) = \begin{cases} \frac{A_1\beta_1^k + A_2\beta_2^k}{\lambda^k + \zeta(1-\lambda)^k} & \text{if } k < k^* - 1 \\ \frac{\mu}{r+\mu}\Big[\lambda U^H p(k) - (\lambda p(k) + (1-\lambda) & \\ \quad \times (1-p(k)))I + \lambda(1-\lambda)\frac{A_1\beta_1^{k-1} + A_2\beta_2^{k-1}}{\lambda^k + \zeta(1-\lambda)^k}\Big] & \text{if } k^* - 1 \leq k < k^* \\ U^H p(k) - I & \text{if } k \geq k^*. \end{cases} \tag{23}$$

To determine the critical level k^*, we use the following boundary conditions. First of all, it should hold that if the number of l-signals relative to the number of h-signals tends to infinity, then the value for the monopolist should converge to zero, i.e.

$$\lim_{k \to -\infty} V(k) = 0.$$

This implies that we only need to consider the larger root β_1.[7] Hence, $A_2 = 0$. To determine A_1 and k^* we solve the continuity condition $V_1(k^* - 1) = V_2(k^* - 1)$ and the value-matching condition $V_2(k^*) = \Omega(k^*)$. Solving the latter equation for A_1 yields

$$\begin{aligned} A_1 = \frac{1}{\beta_1^{k^*-1}\mu\lambda(1-\lambda)} & [U^H \lambda^{k^*}(r + \mu(1-\lambda)) \\ & - rI(\lambda^{k^*} + \zeta(1-\lambda)^{k^*}) - \mu I(\lambda\zeta(1-\lambda)^{k^*} + (1-\lambda)\lambda^{k^*})]. \end{aligned} \tag{24}$$

Substituting A_1 in the former equation yields an expression for $p^* \equiv p(k^*)$:

$$p^* = \frac{1}{\Psi(U^H/I - 1) + 1}, \tag{25}$$

where

$$\Psi = \frac{\beta_1(r+\mu)(r+\mu(1-\lambda)) - \mu\lambda(1-\lambda)(r+\mu(1+\beta_1-\lambda))}{\beta_1(r+\mu)(r+\mu\lambda) - \mu\lambda(1-\lambda)(r+\mu(\beta_1+\lambda))}.$$

The optimal number of h-signals relative to l-signals is then given by

$$k^* = \frac{\log(\frac{p^*}{1-p^*}) + \log(\zeta)}{\log(\frac{\lambda}{1-\lambda})}. \tag{26}$$

[7] This stems from the fact that $\beta_2 < 1-\lambda$, so in $V_1(k)$ and $V_2(k)$ the term β_2^k dominates if $k \to -\infty$ compared with $(1-\lambda)^k$. Hence, if $A_2 \neq 0$, then $V(k) \to \infty$ if $k \to -\infty$.

From eq. (26) it is obtained that k^* decreases with p_0. Hence, less additional information is needed when the initial belief in high revenues is already high.

Next, we check whether the optimal belief p^* is a well-defined probability. The following proposition establishes this result, which is proved in the appendix. It furthermore shows the link between the optimal approach and the traditional NPV rule.

Proposition 3.1 *For $(U^H)^* \geq I$ it holds that $p^* \leq 1$. Furthermore, $p^* > p_{NPV}$.*

So, when the NPV rule is used, the monopolist will invest too soon. It is optimal to wait somewhat longer for more information to arive that reduces uncertainty.

For convenience we summarize the result of the optimal approach in the following theorem.

Theorem 3.1 *The set of combinations of number of signals (n) and h-signals (g), for which it is optimal to invest is given by*

$$\mathcal{B} := \{(n, g) \in \mathbb{N} \times \mathbb{N} | g \leq n, 2g - n \geq \lfloor k^* \rfloor + 1\}. \tag{27}$$

Using eq. (25), one can obtain comparative static results. These are stated in the following proposition, the proof of which is given in the appendix.

Proposition 3.2 *The optimal belief for investment, p^*, increases with I, r and λ and decreases with U^H. It increases with μ iff*

$$\sqrt{(\tfrac{r}{\mu} + 1)^2 - 4\lambda(1-\lambda)} \geq \frac{r(r+\mu)^2 - \mu^3\left((\frac{r}{\mu}+1)^2 - 4\lambda(1-\lambda)\right)}{\mu(\mu^2 - r^2) - 2\mu^3(1-\lambda)}. \tag{28}$$

Condition (28) does not always hold. Simulations show that the condition holds everywhere, except when $\frac{r^2}{\mu^2}$ is close to $2\lambda - 1$, i.e. if $r \approx \mu\sqrt{\lambda - (1-\lambda)}$. Then the denominator in the rhs of eq (28) is close to zero. The rhs of the approximation can be seen as a measure of the usefulness of information in a unit of time, since it is the expected number of signals per time unit times the probability of a correct signal minus the probability of an incorrect signal. So, the condition is not satisfied if the discount rate is approximately the same as the informativeness of the signals.

An important question the monopolist faces is how likely it is that he makes a wrong decision, in the sense that he invests while the market is bad. This question can be answered quantitatively by calculating the probability that k^* is reached while revenues are low. In order to do so, define

$$P^{(k)} := \Pr(\{k^* \text{ is reached starting from } k, \\ \text{given that revenues are low}\}). \tag{29}$$

A second order linear difference equation can now be obtained governing $P^{(k)}$:

$$P^{(k)} = (1-\lambda)P^{(k+1)} + \lambda P^{(k-1)}. \tag{30}$$

Using the boundary conditions

$$P^{(k^*)} = 1$$

and

$$\lim_{k \to -\infty} P^{(k)} = 0,$$

one can solve eq. (30). This yields

$$P^{(k)} = \left(\frac{\lambda}{1-\lambda}\right)^{k-k^*}. \tag{31}$$

Hence, the probability of a wrong decision decreases when the quality of the signals increases. The *ex ante* probability of a wrong decision is given by $P^{(k_0)}$.

4 Economic Interpretation

To make a sensible economic interpretation we must distinguish two cases, namely

1. $I > (U^H)^*$
2. $I \leq (U^H)^*$

In the first case, even if market demand is high the monopolist will make a loss. Therefore, the monopolist will never invest, even if he believes that market demand is high for sure.

The more interesting case is the second one. Now there is an option value of waiting for more information. Let us consider an example where a monopolist faces the problem whether or not to introduce a new product to the market. Market demand can be either positive if the market is good or zero if the market is bad. In the standard microeconomic and Industrial Organization literature this is not considered to be a problem, since the NPV approach is used. Hence, the option value of waiting for more information is not included. The strategic aspects of investing are dealt with quite extensively (cf. Mas-Colell et al. (1995) and Tirole (1988)). Here too, option values are ignored. In more recent contributions this option value is included in the analysis of strategic timing of investment (cf. Hoppe (2000)) and Huisman (2000)).

Inverse demand is assumed to be given by the following function,

$$P(q) = \begin{cases} Y - q & \text{if } q \leq Y \text{ and } H \\ 0 & \text{otherwise,} \end{cases} \tag{32}$$

where q is the quantity supplied. The costs of producing q units are given by the cost function

$$C(q) = cq, \ c \geq 0. \tag{33}$$

The profit of producing q units is then given by

$$\pi(q) = P(q)q - C(q). \tag{34}$$

Suppose for a moment that the market market is good. Then the revenue to the monopolist is given by,

$$\begin{aligned} R_g &= \max_q \left\{ \int_0^\infty e^{-rt}\pi(q)dt \right\} \\ &= \max_q \{\pi(q)\tfrac{1}{r}\}. \end{aligned} \tag{35}$$

Solving for q using the first order condition yields the optimal output level $q^* = \frac{Y-c}{2}$, leading to the optimal revenue

$$U^H = \tfrac{1}{r}[P(q^*)q^* - C(q^*)]. \tag{36}$$

If the market is bad it is optimal not to produce at all. Hence, the revenue if demand is zero, U^L, is given by,

$$U^L = 0. \tag{37}$$

$Y = 8$	$r = 0.1$
$c = 5$	$\mu = 4$
$I = 12$	$\lambda = 0.8$
$p_0 = \frac{1}{2}$	

Table 2: Parameter values

In this example we assume that the monopolist has no *a priori* reason to suspect the market to be good or bad. Hence, it holds that $p_0 = \frac{1}{2}$.

The chosen parameter values are listed in Table 2. So, the discount rate r is set at 10%. Interpreting r are as the riskfree interest rate, this is a common value in the literature. The probability of a correct signal is 0.8 and on average four signals arrive every year.

Based on these parameter values the value function is calculated as function of k and depicted in Figure 2. From this figure one can see that the NPV rule prescribes not to invest at the moment the option becomes available ($k = 0$). In fact, in order to invest, the NPV rule demands that the belief of the monopolist in high market demand should at least be approximately 0.53 ($k_{NPV} \approx 0.10$). However, the optimal approach specifies that the monopolist's belief should exceed $p^* \approx 0.96$ ($k^* \approx 2.23$). This implies that from eq. (9) it can be derived that three h-signals should arrive before it is optimal to invest, provided that no l-signals occur. Hence, the monopolist then should on average wait three quarters of a year before entering the market. The NPV rule prescribes that, in absence of l-signals, only one h-signal is needed. So, the optimal approach prescribes to wait longer than the NPV rule. The difference may be huge. Furthermore, from eq. (31) it is obtained that the probability of taking a wrong decision using the optimal approach equals $P^{(0)} = 0.00156$. Using the NPV rule implies a probability of 0.25. Hence, the probability of making a wrong decision using the optimal approach is much lower than when the NPV rule is used.

Using the same parameters we can see how the critical value k^* changes with values for r, μ, and λ.[8] The case when r is varied is depicted in Figure 3. It shows that if the riskfree rate increases, the critical level rises. This is the so-called present value effect. If r increases future income is valued less

[8] One can see that p^* is not influenced by p_0, although k^* is. We already saw that the latter quantity decreses in p_0.

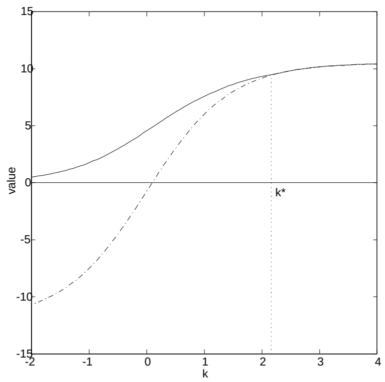

Figure 2: Value function. The dashed line denotes the NPV.

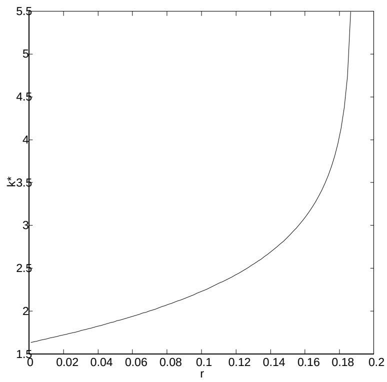

Figure 3: Comparative statics for r

so that the net present value decreases. Therefore, the monopolist is willing to wait longer with investment until he has more certainty about the actual situation in the market. The asymptotic behaviour occurs because of the bound on r that is implicitly given by eqs. (36) and (8).

In Figure 4 we can see that k^* increases with μ.[9] This is because if signals arrive more frequently, it takes less time to get information on the actual market conditions. Therefore, the monopolist will wait for more signals to arrive. If μ tends to infinity, i.e. if the interarrival time between signals converges to zero, then the critical value p^* tends to one, since with so many

[9]Note that eq. (28) is satisfied.

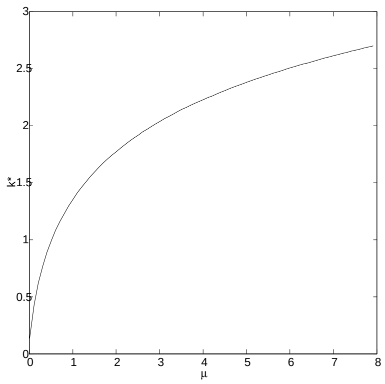

Figure 4: Comparative statics for μ

signals arriving immediately, the uncertainty disappears immediately.[10]

Finally, from Figure 5 we can conclude that the critical level for the believe in high revenues increases with the quality of the signal λ. If λ is

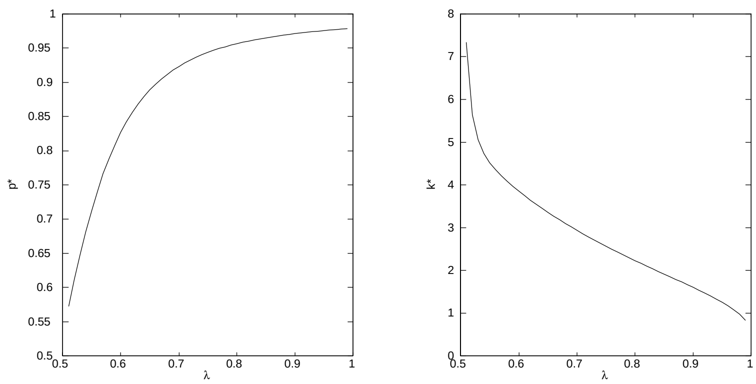

Figure 5: Comparative statics for λ

higher, then the informativeness of a signal is higher. So, it is more attractive for the monopolist to demand a higher certainty about the goodness of the

[10] This can be shown formally by noting that

$$\begin{aligned} n(\Psi) < d(\Psi) = & \mu^2\lambda^2(\beta_1 - (1-\lambda)) + \mu(\beta_1 r + \\ & + r\lambda(\beta_1 - (1-\lambda))) + \beta_1 r^2 \to \infty \text{ if } \mu \to \infty, \end{aligned}$$

where $n(\Psi)$ is the numerator and $d(\Psi)$ is the denominator of Ψ. Hence,

$$\lim_{\mu \to \infty} \Psi = 0, \tag{38}$$

which implies that $\lim_{\mu \to \infty} p^* = 1$.

market. This belief however, is reached after fewer signals as can be seen from the right plane of Figure 5.

5 Conclusions

In this paper a situation was analysed where a monopolist has the opportunity to enter a new market. Market conditions are uncertain, but as time passes the monopolist receives signals about the profitability of the investment. It was shown that the traditional net present value rule fails to reflect the option value that is created by the opportunity to wait for more signals to arrive. The present paper differs from the standard literature on investment under uncertainty (see Dixit and Pindyck (1994)) in that uncertainty resolves in the course of time, which seems a realistic assumption in every day life.

An interesting extension of the present model is to look at what happens when the possible entrant is not a monopolist, but if there are rivalling firms to enter the market. This would require using game theoretic concepts in the present setting. Another topic for further research might be to include costs for receiving the signals. In this way one obtain a model for optimal sampling, closely related to statistical decision theory. Finally, one could extend the idea of resolving uncertainty. For instance, to look at a market where two firms are competing, with imperfect information. Gradually, firms receive signals on each other's behaviour which could then influence their strategies. The latter type of analysis could gain more insight on how competition works in an uncertain environment, which could for instance be applied to antitrust problems. Since the approach in Dixit and Pindyck (1994) does not take into account vanishing uncertainty, one could suspect that this approach puts too much emphasis on the role of uncertainty. This could lead to a *laissez-faire* policy that is taken too far, since one would overestimate e.g. the price margin that should be allowed due to uncertainty. Using the NPV approach would lead to an over-active antitrust policy, as is rightly pointed out in Dixit and Pindyck (1994, ch. 9), since this approach does not take into account any reflection of uncertainty in the price margin. Applying the ideas presented in this paper to an established market analysis we would expect that the mark-up of uncertainty in the price margin in a certain industry depends critically on the level of maturity in the industry.

A Proof of Proposition 3.1

Denote the denominator of Ψ by $d(\Psi)$. Analogously, we denote the numerator of Ψ by $n(\Psi)$. Note that (using $\beta_1 \geq 1-\lambda$)

$$\begin{aligned} n(\Psi) &= \beta_1(r+\mu)(r+\mu\lambda) - \mu\lambda(1-\lambda)(r+\mu(\beta_1+\lambda)) \\ &\quad - \mu(2\lambda-1)[\beta_1(r+\mu) - \mu\lambda(1-\lambda)] \\ &\leq \beta_1(r+\mu)(r+\mu\lambda) - \mu\lambda(1-\lambda)(r+\mu(\beta_1+\lambda)) \\ &\quad - \mu(2\lambda-1)[r(1-\lambda)+\mu(1-\lambda)^2] \\ &< d(\Psi). \end{aligned} \tag{39}$$

Hence, $\Psi < 1$. If $r = 0$, it holds that $\beta_1 = \lambda$. Therefore,

$$n(0) = \lambda\mu^2(1-\lambda) - \mu\lambda(1-\lambda)\mu = 0.$$

Furthermore, using that $\beta_1 \geq (1-\lambda)$, it can be obtained that

$$\begin{aligned} \frac{dn(\Psi)}{dr} &= 2\beta_1 r - \mu\lambda(1-\lambda) + \beta_1\mu(2-\lambda) \\ &\quad + \frac{\partial\beta_1}{\partial r}\left(r(\mu(2-\lambda)+r) + \mu^2(1-\lambda)^2\right) \\ &\geq 2r(1-\lambda) + 2\mu(1-\lambda)^2 + \frac{\partial\beta_1}{\partial r}\left(r(\mu(2-\lambda)+r) + \mu^2(1-\lambda)^2\right) \\ &> 0, \end{aligned} \tag{40}$$

since $\frac{\partial\beta_1}{\partial r} > 0$. So, $\Psi > 0$ and p^* is a well-defined probability. Furthermore, since $\Psi < 1$, it holds that

$$\begin{aligned} p^* &= \frac{1}{\Psi(\frac{(U^H)^*}{I} - 1) + 1} \\ &> \frac{I}{(U^H)^*} = p_{NPV}. \end{aligned}$$

□

B Proof of Proposition 3.2

Simple calculus gives the result for U^H and I. To prove the proposition for r, μ, and λ, let us first derive the comparative statics of β_1 for these parameters. First, take r. The total differential of $\mathcal{Q}$ with respect to r is given by

$$\frac{\partial \mathcal{Q}}{\partial \beta_1}\frac{\partial \beta_1}{\partial r} + \frac{\partial \mathcal{Q}}{\partial r} = 0.$$

From Figure 1 one can see that $\frac{\partial \mathcal{Q}}{\partial \beta_1} > 0$. Furthermore, $\frac{\partial \mathcal{Q}}{\partial r} = -\frac{\beta_1}{r} < 0$. Hence, it must hold that $\frac{\partial \beta_1}{\partial r} > 0$. In a similar way one obtains $\frac{\partial \beta_1}{\partial \mu} < 0$ and $\frac{\partial \beta_1}{\partial \lambda} > 0$.

From the proof of Proposition 3.1 one can see that the numerator and denominator of Ψ can be written in the following form

$$\begin{aligned} n(\Psi) &= \eta(r,\mu,\lambda) - 2\mu(1-\lambda)\zeta(r,\mu,\lambda), \\ d(\Psi) &= \eta(r,\mu,\lambda) - 2\mu(1-\lambda)\nu(r,\mu,\lambda), \end{aligned}$$

where

$$\begin{aligned} \eta(r,\mu,\lambda) &= \beta_1(r+\mu)(r+\mu\lambda) - \mu\lambda(1-\lambda)(r+\mu(\beta_1+\lambda)), \\ \zeta(r,\mu,\lambda) &= \beta_1(r+\mu) - \mu\lambda(1-\lambda), \\ \nu(r,\mu,\lambda) &= r(1-\lambda) + \mu(1-\lambda)^2. \end{aligned}$$

Since $\Psi > 0$, this implies that to determine the sign of the derivative of Ψ with respect to one of the parameters, one only needs to compare the respective derivatives of $\zeta(\cdot)$ and $\nu(\cdot)$. Note that

$$\begin{aligned} \frac{\partial \zeta(\cdot)}{\partial r} &= \beta_1 + \frac{\partial \beta_1}{\partial r} r > \beta_1 \\ &\geq 1 - \lambda = \frac{\partial \nu(\cdot)}{\partial r}. \end{aligned} \tag{41}$$

Hence, $\frac{\partial \Psi}{\partial r} < 0$ and $\frac{\partial p^*}{\partial r} > 0$.

For λ a similar exercise can be done, yielding

$$\begin{aligned}\frac{\partial \zeta(\cdot)}{\partial \lambda} &= \mu(2\lambda - 1) + (r+\mu)\frac{\partial \beta_1}{\partial \lambda} > 0 \\ &> -\big(r + 2\mu(1-\lambda)\big) = \frac{\partial \nu(\cdot)}{\partial \lambda}.\end{aligned} \tag{42}$$

Hence, $\frac{\partial p^*}{\partial \lambda} > 0$.

To prove the result on μ, one needs to calculate $\frac{\partial \beta_1}{\partial \mu}$ explicitly. This yields

$$\frac{\partial \beta_1}{\partial \mu} = -r\frac{\beta_1}{\mu^2\sqrt{(\frac{r}{\mu}+1)^2 - 4\lambda(1-\lambda)}}. \tag{43}$$

Furthermore, using this result, one can show

$$\begin{aligned}\frac{\partial \zeta(\cdot)}{\partial \mu} &= \frac{\partial \beta_1}{\partial \mu}(r+\mu) + \beta_1 - \lambda(1-\lambda) \\ &= \beta_1\frac{\mu^2\sqrt{(\frac{r}{\mu}+1)^2 - 4\lambda(1-\lambda)} - r(r+\mu)}{\mu^2\sqrt{(\frac{r}{\mu}+1)^2 - 4\lambda(1-\lambda)}} - \lambda(1-\lambda), \\ \frac{\partial \nu(\cdot)}{\partial \mu} &= (1-\lambda)^2.\end{aligned}$$

Hence, substituting for β_1 using eq. (18), one obtains

$$\begin{aligned}&\frac{\partial \zeta(\cdot)}{\partial \mu} \geq \frac{\partial \nu(\cdot)}{\partial \mu} \\ \Leftrightarrow&\Big(\frac{r+\mu}{2\mu} + \tfrac{1}{2}\sqrt{(\tfrac{r}{\mu}+1)^2 - 4\lambda(1-\lambda)}\Big)\frac{\mu^2\sqrt{(\frac{r}{\mu}+1)^2 - 4\lambda(1-\lambda)} - r(r+\mu)}{\mu^2\sqrt{(\frac{r}{\mu}+1)^2 - 4\lambda(1-\lambda)}} \\ &\geq 1-\lambda \\ \Leftrightarrow&\big(\mu^2(r+\mu) - \mu r(r+\mu)\big)\sqrt{(\tfrac{r}{\mu}+1)^2 - 4\lambda(1-\lambda)} - r(r+\mu)^2 \\ &+\mu^3\big((\tfrac{r}{\mu}+1)^2 - 4\lambda(1-\lambda)\big) \geq 2\mu^3(1-\lambda)\sqrt{(\tfrac{r}{\mu}+1)^2 - 4\lambda(1-\lambda)} \\ \Leftrightarrow&\sqrt{(\tfrac{r}{\mu}+1)^2 - 4\lambda(1-\lambda)} \geq \frac{r(r+\mu)^2 - \mu^3\big((\frac{r}{\mu}+1)^2 - 4\lambda(1-\lambda)\big)}{\mu(\mu^2 - r^2) - 2\mu^3(1-\lambda)}.\end{aligned}$$

The last inequality is precisely eq. (28). So, then $\frac{\partial p^*}{\partial \mu} > 0$. □

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